Note on completeness and its consequences for Week 9

Definition 1. *If $X \subseteq \mathbb{R}$ then X is said to be bounded if there is some $b \in \mathbb{R}$ such that $x \leq b$ for all $x \in X$. The real number b is said to be an upper bound for X.*

Additional exercise 1. *If a set $X \subseteq \mathbb{R}$ is bounded then there are infinitely many upper bounds for X.*

Definition 2. *If $X \subseteq \mathbb{R}$ and $b \in \mathbb{R}$ then b is said to be the supremum of X if*

(1) *b is an upper bound for X*
(2) *if $\bar{b} < b$ then $\bar{b}$ is not an upper bound for X.*

Additional exercise 2. *A set $X \subseteq \mathbb{R}$ has at most one supremum.*

Theorem 1. *Every non-empty, bounded set of reals has a supremum.*

Proof. This will be proved for $X \subseteq [0, 1]$ since the general case follows easily from this. Each $x \in X$ can be represented in decimal expansion as an infinite sequence $x = 0.x_1x_2x_3\ldots$ of the ten digits $0, 1, 2, 3\ldots, 8, 9$. In other words, each x_n is an integer between 0 and 9.

Let s_1 be the largest of all the digits x_1 where $x \in X$. Then let s_2 be the largest of al the digits x_2 where $x \in X$ and $x_1 = s_1$. Proceeding by induction, let s_{n+1} be the largest of all the digits x_{n+1} where:

- $x \in X$
- $x_1 = s_1$
- $x_2 = s_2$
- $\vdots$
- $x_n = s_n$

Then let s be the real number whose digits in decimal expansion are the s_i; in other words,

$$s = 0.s_1s_2s_3s_4\ldots$$

The first thing to show is that s is an upper bound for X; namely that if $x \in X$ then $x_n \leq s_n$ for each natural number n. So let $\bar{x} \in X$. It will be shown by induction that $\bar{x}_n \leq s_n$. If $n = 1$ then s_1 was defined to be the largest of all x_1 where $x \in X$. Since $\bar{x} \in X$ it follows that $\bar{x}_1 \leq s_1$. Now suppose that it has been shown that $\bar{x}_n \leq s_n$ for all $n \leq m$. Then s_{n+1} was defined to be the largest of all the digits x_{n+1} where:

- $x \in X$
- $x_1 = s_1$
- $x_2 = s_2$
- $\vdots$
- $x_n = s_n$

In particular, $\bar{x}$ satisfies all of the conditions and hence $\bar{x}_{n+1} \leq s_{n+1}$.

Finally it must be shown that if $s' < s$ then s' is not an upper bound for X. But if $s' < s$ there must be some digit where s' and s disagree. Let n be the least such digit; in other words,

- $s'_n \neq s_n$
- $s'_1 = s_1$
- $s'_2 = s_2$
- $\vdots$
- $s'_{n-1} = s_{n-1}$

Moreover, since $s' < s$ it must be the case that $s'_n < s_n$. But by the definiton of s_n there must be some $x \in X$ such that

- $x_1 = s_1$
- $x_2 = s_2$

- $\vdots$
- $x_{n-1} = s_{n-1}$

and $x_n = s_n$. But then the first digit where s' and x disagree is the n^{th} digit and $x_n > s'_n$, meaning that $s' < x$. This shows that s' is not an upper bound for X. $\square$

Additional exercise 3. *Finish the proof that bounded sets of reals have suprema; in other words, complete the proof for X that are not necessarily subsets of $[0,1]$.*

Theorem 2 (Intermediate Value Theorem)**.** *If If f is a continuous function from $[a,b]$ to $\mathbb{R}$ and $f(a) < v < f(b)$ then there is $x \in [a,b]$ such that $f(x) = v$.*

Proof. Suppose that $f : [a,b] \to \mathbb{R}$ is continuous and $f(a) < v < f(b)$. Let X be the set of $y \in [a,b]$ such that $f(z) \leq v$ for all $z \in [a,y]$. Since $a \in X$ it follows that X is non-empty and bounded. Letting x be the supremum of X it suffices to show that $f(x) = A$. This will be proved by contradiction by considering two cases.

Consider first the case that $f(x) > v$. Let $\epsilon = f(x) - v > 0$. The continuity of f at x yields $\delta > 0$ such that $f(w) > v$ for all w such that $x - \delta < w < x + \delta$. In other words, the interval $(x - \delta, x + \delta)$ is disjoint from X and this implies that $x - \delta$ is an upper bound for X, contradicting that x is the supremum of X.

Consider next the case that $f(x) < v$. From the hypothesis that $f(a) < v < f(b)$ it follows that $x < b$. Let $\epsilon = v - f(x) > 0$. The continuity of f at x yields $\delta > 0$ such that $f(w) < v$ for all w such that $x - \delta < w < x + \delta$. This implies that $x + \delta \in X$ contradicting that x is an upper bound for X.

Since both cases lead to a contradiction, it follows that $f(x) = v$.

$\square$

Theorem 3 (Extreme Value Theorem)**.** *If f is a continuous function from $[a,b]$ to $\mathbb{R}$ then there is some B such that:*

- *$f(x) \leq B$ if $a \leq x \leq b$*
- *there is some $x \in [a,b]$ such that $f(x) = B$.*

Proof. A very similar proof to the case of Intermediate Value Theorem works here. Suppose that $f : [a,b] \to \mathbb{R}$ is continuous. To see that f is bounded, let X be the set of all $y \in [a,b]$ such that there is some B such that $f(z) < B$ for all $z \in [a,y]$. It will be shown by contradiction that b is the supremum of X.

Suppose that $s < b$ and s is the supremum of X. By continuity, letting $\epsilon = 1$, there is some $\delta > 0$ such that $f(x) < f(s) + 1$ for all x such that $s - \delta < x < s + \delta$. Let $x^* \in X$ be arbitrary such that $s - \delta < x^* < s$. This means that there must be some B be such that $f(z) < B$ for all $z \in [a, x^*]$. But then $f(z) < \max(B, f(s) + 1)$ for all z such that $z < s + \delta$. This contradicts that s is an upper bound for X since every real in $(s, s + \delta)$ is also in X. Hence $s = b$.

Finally it must be shown that $b \in X$. This is the same as the earlier argument. By continuity, letting $\epsilon = 1$, there is some $\delta > 0$ such that $f(x) < f(b) + 1$ for all x such that $s - \delta < x \leq b$. Let $x^* \in X$ be arbitrary such that $s - \delta < x^* < s$. This means that there must be some B be such that $f(z) < B$ for all $z \in [a, x^*]$. But then $f(z) < \max(B, f(b) + 1)$ for all z such that $z \leq b$.

Now let B be the supremum of the range of f. (Why does this exist?) Then let Y be the set of $z \in [a,b]$ such that the supremum of the range of f on $[a,z]$ is less than BR and let x be the supremum of Y. We now have to check that $f(x) = B$. If $f(x) < B$ then a contradiction is obtained just as in the proof of the Intermediate Value Theorem. In other words, it must be the case that $f(x) = B$. $\square$

Additional exercise 4. *Define a function f to be strongly continuous if for all x in the domain of f there is $\delta > 0$ such that if $|z - x| < \delta$ then $|f(z) - f(x)| = 0$. Prove that if the domain of f is the interval $[a,b]$ and f is strongly continuous then f is constant.*

Additional exercise 5. *Define a function to be locally increasing if for each x in its domain there is $\delta > 0$ such that the function is increasing on the interval $(x - \delta, x + \delta)$. Prove that if the domain of f is the interval $[a, b]$ and f is locally increasing then f is increasing.*

Additional exercise 6. *Define a function to be locally monotone if for each x in its domain there is $\delta > 0$ such that the function is increasing on the interval $(x - \delta, x + \delta)$ or decreasing on the interval $(x - \delta, x + \delta)$. Prove that if the domain of f is the interval $[a, b]$ and f is locally monotone then f is monotone. Be careful, since this does not follow immediately from the previous exercise.*

Additional exercise 7. *Suppose that*

- $X \subseteq (a, b)$
- *b is the supremum of X*
- *$-a$ is the supremum of $-X$ where $-X$ is defined to be the set of all $-x$ where $x \in X$.*

Prove that the supremum of the $X - X$ is $b - a$ where $X - X$ is defined to be the set of all $x - y$ where x and y beong to X.

Additional exercise 8. *Suppose that $f : [0, 1] \to \mathbb{R}$ is continuous and $f(0) = f(1) = 0$. Suppose further that $f(a) > 0$. Prove that there is some b such that:*

- $1 \leq b < a$
- $f(b) = 0$
- *$f(x) > 0$ for all x such that $b < x \leq a$.*

Additional exercise 9. *Suppose that $f : [0, 1] \to \mathbb{R}$ is increasing (or even just non-decreasing) on $[0, 1]$ and $0 < a < 1$. Prove that $\lim_{x \to a^-} f(x)$ exists.*